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Tilting the noncommutative bion

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Abstract: In this note, we extend the noncommutative bion core solution of Constable, Myers and Tafjord [4] to include the effects of a nonzero NS-NS two-form B. The result is a 'tilted bion', in which the core expands out to a single D3-brane at an angle to the D1-brane core. Its properties agree perfectly with an analysis of the dual situation, that of a magnetic charge on an abelian D3-brane in a background worldvolume magnetic field. We also demonstrate that this agreement extends beyond geometry to include the field strength on the D3-brane. We make a proposal for including possible worldvolume gauge fields when mapping a noncommutative geometrical brane solution onto a corresponding commutative brane description.

Keywords: Brane Dynamics in Gauge Theories, Non-Commutative Geometry.

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Contents

1. Introduction

Much effort has been devoted to generalizing the Born–Infeld action so as to describe the dynamics of multiple superposed D-branes. It is known that the gauge group for a stack of N superposed D-branes is enhanced from $\mathrm{U}(1)^N$ to $\mathrm{U}(N)$ and that the brane worldvolume supports a $\mathrm{U}(N)$ gauge field as well as a set of scalars in the adjoint representation of $\mathrm{U}(N)$ (one for each of the transverse coordinates). A specific proposal for a generalized action involving such fields has recently been given by Myers in [1] (see also references in this paper for other work on this problem). For a Dp-brane in static gauge, Myers' action takes the following, rather forbidding, form:

$$S_{BI} = -T_p \int d^{p+1}\sigma \,\mathrm{STr} \times \tag{1.1}$$
$$\times \left(e^{-\phi} \sqrt{-\det(P_{ab}\left[(G+\lambda B)_{\mu\nu} + (G+\lambda B)_{\mu i}(Q^{-1}-\delta)^{ij}(G+\lambda B)_{j\nu}\right] + \lambda \mathcal{F}_{ab})\det(Q^i_j)} \right),$$

where

$$Q^i_j \equiv \delta^i_j + i\lambda \left[\Phi^i, \Phi^k\right](G+\lambda B)_{kj}\,, \tag{1.2}$$

$\lambda \equiv 2\pi\alpha'$, a, b are indices in the worldvolume of the Dp-branes, i, j, k are in the transverse space and μ, ν are ten-dimensional indices. The Φ^i are $N \times N$ matrix scalar fields describing the transverse displacements of the branes and $\mathcal{F}_{ab}$ is the worldvolume gauge field (both fields are in the adjoint representation of $\mathrm{U}(N)$). The symbol $P_{ab}[M]$ stands for a pullback of the ten-dimensional matrix M to the brane worldvolume in which the matrix coordinates Φ^i define the surface to which to pull back and all derivatives of these coordinates are made gauge covariant. Finally, STr is Tseytlin's symmetrized trace operation [2]. We refer the reader to Myers' paper [1] for a more complete definition and a discussion of useful simplifying approximations. Whether or not this action is "exact", it does seem to capture a lot of information about such structures as the commutators of the Φ's with themselves, which vanish in the U(1) limit and cannot be directly inferred from the abelian Born-Infeld action.

In [4], Myers' action was applied to a stack of N D1-branes in a flat background and it was shown that the transverse coordinates of the D1-branes 'flare-out' to a flat three-dimensional space. This was interpreted as a description of a collection of D1-branes attached to an orthogonal D3-brane, one of the standard D-brane intersections. This situation has a 'dual' description in terms of a single (abelian) D3-brane carrying a point magnetic charge. Using the Born-Infeld action, one finds that a magnetic monopole of strength N produces a singularity in the D3-brane's transverse displacement, a 'spike' which can be interpreted as N D1-strings attached to the D3-brane [5, 6]. To the extent that it is possible to compare them, there is a perfect quantitative match between the two pictures. The states in question are BPS states and this match is evidence that the Myers' action (1.1) captures the dynamics of BPS states at least.

In this brief note, we extend these considerations to a more complex example of a BPS state: we generalize the solution of [4] to include a nonzero background NS-NS two-form field B, in a direction transverse to the D1-branes, but parallel to the emergent D3-brane. In the 'dual' description on the D3-brane, the B field becomes a U(1) magnetic field on the D3-brane which pulls on the magnetic charge associated with the D1-brane. Simple force balance considerations suggest that the spike representing the D1-branes should be tilted away from the direction normal to the D3-brane, and this is precisely what is found in explicit solutions of the Born-Infeld equations [9, 10]. We will show that this system can be analysed in the nonabelian D1-brane picture and that there is, once again, a perfect quantitative match between the results of the two dual calculations. One slight novelty of our approach is that we can demonstrate this agreement not only for geometrical quantities, but also for worldvolume gauge fields.

The paper is organized as follows: In section 2, we review how the bion arises in the abelian theory on a D3-brane, both with and without a B-field. In section 3, we review the construction of the nonabelian bion, and then extend it to a nonzero B-field. In section 4, we take a brief detour and describe how lower-dimensional D-branes can form flat, noncommutative, higher dimensional D-branes equipped with worldvolume gauge fields. In this context we show how the higher-dimensional worldvolume gauge field is constructed out of the lower-dimensional noncommuting coordinates. Finally, in section 5, we apply this recipe to the gauge field on the bion of section 3 and show that there is perfect agreement between the two approaches, in both the geometry and the field strength on the brane.

2. The bion solution on a D3-brane

Consider the abelian Born-Infeld action for a single D3-brane in flat space. Let the D3-brane extend in 0123-directions, and let the coordinates on the brane be denoted by x^i, $i = 0, \ldots, 3$. Restricting the brane to have displacement in only one of the transverse directions, we can take the (static gauge) embedding coordinates of the brane in the ten-

JHEP05(2002)038

dimensional space to be $X^i = x^i$, $i = 0, \ldots, 3$; $X^a = 0$, $a = 4, \ldots, 8$; $X^9 = \sigma(x^i)$. Then there exists a static (BPS) solution of the Born-Infeld action corresponding to placing N units of U(1) magnetic charge at the origin of coordinates on the brane [5]:

$$X^9(x^i) = \sigma(x^i) = \frac{q}{\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}}\,, \tag{2.1}$$

where $q = \pi\alpha' N$, and N is an integer. This magnetic bion solution to the Born–Infeld action corresponds to N superposed D1-strings attached to the D3-brane at the origin. It is "reliable" in the sense that the effect of unknown higher-order corrections in α' and g to the action can be made systematically small in the large-N limit (see [5] for details). At a fixed $X^9 = \sigma$, the cross-section of the deformed D3-brane is a 2-sphere with a radius

$$r(\sigma) = \frac{\pi\alpha' N}{\sigma}\,. \tag{2.2}$$

In the presence of a two-form field B parallel to the world volume of the brane, ($\frac{1}{2}B dx^1 \wedge dx^2$ to be concrete), the above solution is modified as follows [10]:

$$\frac{\sigma}{\cos(\alpha)} = \frac{q}{\sqrt{(x^1)^2 + (x^2)^2 + \cos(\alpha)^2 (x^3 - \tan(\alpha)\sigma)^2}}\,. \tag{2.3}$$

where $\tan(\alpha) = 2\pi\alpha' B$. Because the transverse displacement σ is not a single-valued function of the base coordinates x^i, the geometry is somewhat obscure. It is more transparent in rotated coordinates defined by

$$Y^1 = X^1\,, \quad Y^2 = X^2\,, \quad Y^3 = \cos(\alpha)X^3 - \sin(\alpha)X^9\,, \quad Y^4 = \sin(\alpha)X^3 + \cos(\alpha)X^9\,. \tag{2.4}$$

Choosing $Y^{1,2,3}$ as the worldvolume coordinates, the embedding becomes

$$Y^{(1,2,3)}(y) = y^{(1,2,3)}\,, \qquad Y^4(y) = \tan(\alpha)\; y^3 + \frac{q}{\sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}}\,. \tag{2.5}$$

It is easy to see that this describes D1-strings tilted away from the normal to the D3-brane by an angle α (when $B \to 0$, $\alpha \to 0$, and the branes become orthogonal).

Reverting to the original coordinates (2.3), we can show that the D3-brane at a fixed transverse displacement $X^9 = \sigma$ is an ellipsoid of revolution defined by the equation

$$1 = \frac{x_1^2}{r_1^2(\sigma)} + \frac{x_2^2}{r_2^2(\sigma)} + \frac{(x_3 - \sigma\tan(\alpha))^2}{r_3^2(\sigma)} \tag{2.6}$$

with major and minor axes

$$r_1(\sigma) = r_2(\sigma) = \cos(\alpha)\frac{\pi\alpha' N}{\sigma}\,, \qquad r_3(\sigma) = \frac{\pi\alpha' N}{\sigma}\,. \tag{2.7}$$

For large σ, the ellipsoid becomes small and defines a slice through the D1-brane that is attached to the D3-brane. The ellipsoid is centered at brane coordinates $(X_1, X_2, X_3) = (0, 0, \tan(\alpha)\sigma)$ and the fact that X_3 varies linearly with σ implements the tilt of the D1-brane. The tilting arises for simple reasons of force balance. The D1-brane spike behaves like a magnetic charge from the point of view of the worldvolume gauge theory; the background B field is equivalent to a uniform magnetic field on the D3-brane and exerts a force on the magnetic charge which must be balanced by a component of the D1-brane tension along the D3-brane .

The gauge field on the brane is particularly easy to obtain in the tilted coordinates (2.4). The configurations under discussion are not the most general solution of the equations of motion: they are special minimal energy solutions that satisfy the BPS condition (and preserve some supersymmetry). The BPS condition relates the total 2-form field on the brane to the divergence of the transverse displacement scalar as follows [5, 6]:

$$(2\pi\alpha')\epsilon^{ijk}(F+B)_{jk} = \pm\frac{\partial}{\partial y^i}Y^4 = \frac{\mp q}{[(y^1)^2+(y^2)^2+(y^3)^2]^{3/2}}\, y^i + \tan(\alpha)\delta^i_3\,. \tag{2.8}$$

The ambiguous sign encodes the difference between a D3- and a $\overline{\text{D3}}$-brane. From this we read off that the magnetic field $\mathcal{B}_k \equiv \epsilon^{ijk}F_{jk}$ is just

$$\mathcal{B}_i(y) = \mp\frac{N}{2[(y^1)^2+(y^2)^2+(y^3)^2]^{3/2}}\, y^i\,, \tag{2.9}$$

or a Coulomb field due to N charges. The end of the D1-brane(s) acts as a magnetic charge and the space-time B field provides an effective uniform magnetic field which exerts a force on the magnetic charge, hence tilting the D1-branes.

Note that for the upper sign (D3-brane), when N is positive, the D1-brane(s) run 'towards' the D3-brane, and when it is negative, they run 'away' from the D3-brane. For the lower sign ($\overline{\text{D3}}$-brane case), the end of the $\overline{\text{D1}}$-brane(s) running 'towards' the $\overline{\text{D3}}$-brane acts as a positive magnetic source and the end of the $\overline{\text{D1}}$-brane(s) running 'away' from $\overline{\text{D3}}$-brane acts as negative one. We will encounter the same four cases in our dual treatment by the nonabelian D1-branes.

3. Dual treatment by nonabelian D1-branes

In describing the intersection of one D3-brane with N D1-branes, one has the option of starting from the dynamics of the D3-brane and trying to derive the D1-branes (this was the approach of the previous section), or of starting from the nonabelian dynamics of multiple D1-branes and trying to derive the D3-brane. The latter approach has been applied in [4] to the case in which there is no background B field. In this section we will review that work and show how to generalize it to the case where $B \neq 0$ and the bion is tilted.

We begin by reviewing the results from [4] on the $B = 0$ case, while introducing some notation which will be useful later. We consider the nonabelian Born–Infeld action of equation (1.1) specialized to the case of N coincident D1-branes, flat background spacetime ($G_{\mu\nu} = \eta_{\mu\nu}$), vanishing B field, vanishing worldvolume gauge field and constant dilaton. The action then depends only on the $N \times N$ matrix transverse scalar fields Φ^i's. In general, $i = 1, \ldots, 8$, but since we are interested in studying the

D1/D3-brane intersection, we will allow only three transverse coordinate fields to be active ($i = 1, 2, 3$). The explicit reduction of the static gauge action ($X^0 = \tau$ and $X^9 = \sigma$) is then

$$S_{BI} = -T_1 \int d\sigma d\tau \,\mathrm{STr} \sqrt{-\det\left(\eta_{ab} + \lambda^2 \partial_a \Phi^i Q^{-1}_{ij} \partial_b \Phi^j\right) \det\left(Q^{ij}\right)}\,, \tag{3.1}$$

where

$$Q^{ij} = \delta^{ij} + i\lambda[\Phi^i, \Phi^j]\,. \tag{3.2}$$

Since the dilaton is constant, we incorporate it in the tension T_1 as a factor of g^{-1}.

We look for static solutions (i.e., $\Phi = \Phi(\sigma)$ only). Since we have no hope of finding a general static solution of these nonlinear matrix equations, we make some simplifying assumptions which have a chance of being valid on the restricted class of BPS solutions. The action (3.1) depends only on the two matrix structures $\partial_a \Phi^i$ and $W_i \equiv \frac{1}{2} i\epsilon_{ijk}[\Phi^j, \Phi^k]$ and, because of the nature of the STr instruction, they may be treated as commuting quantities until the final step of doing the gauge trace to evaluate the action. This allows us to evaluate the determinants in the definition of the action (3.1) and to convert the energy functional to the following form:

$$U_{B=0} = \int d\sigma \,\mathrm{STr} \sqrt{1 + \lambda^2 (\partial\Phi^i)^2 + \lambda^2 (W_i)^2 + \lambda^4 (\partial\Phi^i W_i)^2}\,. \tag{3.3}$$

Continuing to treat Φ^i and W^i as commuting objects, we see that this energy functional can be written as a sum of two squares

$$U_{B=0} = \int d\sigma \,\mathrm{STr} \sqrt{(1 \pm \lambda^2 \partial\Phi^i W_i)^2 + \lambda^2 (\partial\Phi^i \mp W_i)^2}\,, \tag{3.4}$$

and is minimized by a displacement field satisfying the first-order BPS-like equation $\partial\Phi^i = \pm W_i$. This equation, written more explicitly as

$$\partial\Phi_i = \pm\frac{1}{2} i\epsilon_{ijk}[\Phi^j, \Phi^k]\,, \tag{3.5}$$

is known as the Nahm equation [11]. The $\pm$ ultimately corresponds to the choice between a bundle of D1- or $\overline{\mathrm{D1}}$-branes. The Nahm equation is a very plausible candidate for the exact equation to be satisfied by a BPS solution of this system and the fact that the Myers action (3.1) implies it in the BPS limit is very satisfactory.

The Nahm equation has the trivial solution $\Phi = 0$ which corresponds to an infinitely long bundle of coincident D1-branes. In [4], a much more interesting solution was found by starting with the following ansatz:

$$\Phi^i = \hat{R}(\sigma)\alpha^i\,, \qquad (\alpha^1, \alpha^2, \alpha^3) \equiv \mathbf{X}\,, \tag{3.6}$$

where α^i form an $N \times N$ representation of the generators of an SU(2) subgroup of U(N), $[\alpha^i, \alpha^j] = 2i\epsilon_{ijk}\alpha^k$. With this ansatz, both $\partial\Phi^i$ and W^i are proportional to the generator matrix α^i. When the ansatz is substituted into the BPS condition (3.5), we obtain a simple equation for $\hat{R}$,

$$\hat{R}' = \mp 2\hat{R}^2\,, \tag{3.7}$$

which is solved by

$$\hat{R} = \pm\frac{1}{2\sigma}\,. \tag{3.8}$$

Substituting the ansatz (3.6) into (3.3) leads to the following effective action for $\hat{R}(\sigma)$:

$$U_{B=0}[\hat{R}(\sigma)] = \int d\sigma\, \mathrm{STr}\sqrt{(1+\lambda^2(\hat{R}')^2\mathbf{X}^2)(1+4\lambda^2(\hat{R})^4\mathbf{X}^2)}\,. \tag{3.9}$$

It can be shown that (3.8) satisfies the equations of motion following from the action (3.9).

This solution maps very nicely onto the bion solution of the previous section. At a fixed point $|\sigma|$ on the D1-brane stack, the geometry given by (3.6) is that of a sphere with the physical radius $R^2 = \frac{\lambda^2}{N}Tr(\Phi^i)^2$ (the only sensible way to pass from the matrix transverse displacement field $\lambda\Phi^i$ to a pure number describing the geometry). For the ansatz under consideration, this gives

$$R(\sigma)^2 = \frac{\lambda^2}{N}Tr(\Phi^i)^2 = \lambda^2\hat{R}(\sigma)^2 C\,, \tag{3.10}$$

where C is the quadratic Casimir, equal to N^2-1 for an irreducible representation of SU(2). This gives

$$R(\sigma) = \frac{\lambda\sqrt{N^2-1}}{(2|\sigma|)} \cong \frac{\pi\alpha' N}{|\sigma|} \tag{3.11}$$

for large N, in agreement with equation (2.2). This completes our synopsis of the arguments given in [4] for the agreement between the commutative and noncommutative approaches to the D1/D3-brane intersection.

A simple argument can be made at this point to strengthen the meaning of equation (3.11). At a fixed σ, the Fourier transform of the density of the D1-strings is given by [3]

$$\widetilde{\rho}(k) = Tr\left(e^{i\lambda k_i \Phi^i}\right). \tag{3.12}$$

This is simply the operator to which the 09-component of the RR 2-form $C^{(2)}$, C_{09}, couples. For the solution (3.6), this is evaluated to give

$$\widetilde{\rho}(k) = \frac{\sin(\lambda N\hat{R}|k|)}{\sin(\lambda\hat{R}|k|)}\,, \tag{3.13}$$

which for $N \gg 1$, and k such that $(\lambda N\hat{R})^{-1} < |k| \ll (\lambda\hat{R})^{-1}$ (i.e., for momentum large enough to resolve the size of the sphere, but not large enough to resolve the individual brane constituents) gives

$$\widetilde{\rho}(k) \cong \frac{\sin(\lambda N\hat{R}|k|)}{\lambda\hat{R}|k|}\,. \tag{3.14}$$

This is precisely the Fourier transform of the density distribution representing a thin shell,

$$\widetilde{\rho}(k) = \int d^3x e^{i\vec{k}\cdot\vec{x}}\rho(x) \quad \text{for} \quad \rho(x) = \frac{N}{4\pi R^2}\delta(|x|-R) \quad \text{with} \quad R = \lambda N\hat{R} = \frac{\pi\alpha' N}{|\sigma|}\,, \tag{3.15}$$

in agreement with equation (3.11).

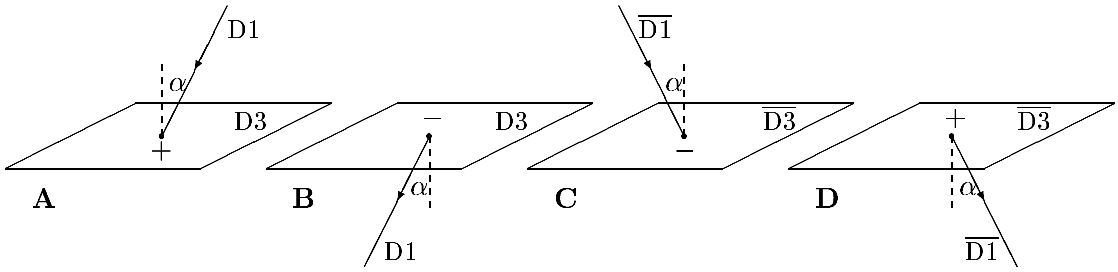

Figure 1: The four cases **A**-**D** discussed in the text.

To distinguish the various cases involving branes and antibranes, note that σ can either run from $-\infty$ to 0, in which case the D1($\overline{\text{D1}}$)-branes run 'towards' the D3($\overline{\text{D3}}$)-brane plane, or from 0 to ∞, in which case the D1($\overline{\text{D1}}$)-branes run 'away' the D3($\overline{\text{D3}}$)-brane plane. We have thus four cases:

A : Stack of D1-branes expanding to a D3-brane, $\partial\Phi^i = +W_i$ and $\sigma \in (-\infty, 0)$

B : Stack of D1-branes expanding to a D3-brane, $\partial\Phi^i = +W_i$ and $\sigma \in (0, \infty)$

C : Stack of $\overline{\text{D1}}$-branes expanding to a $\overline{\text{D3}}$-brane, $\partial\Phi^i = -W_i$ and $\sigma \in (-\infty, 0)$

D : Stack of $\overline{\text{D1}}$-branes expanding to a $\overline{\text{D3}}$-brane, $\partial\Phi^i = -W_i$ and $\sigma \in (0, \infty)$

Cases **A** and **D** correspond to the D1($\overline{\text{D1}}$)-branes running 'towards' ('away from') the D3($\overline{\text{D3}}$)-brane plane, and thus should represent a positive magnetic charge, while cases **B** and **C** should represent a negative magnetic charge. We will see shortly that this is the case and that the four cases match the four cases found in the abelian treatment of the same problem starting from the D3-brane (see the end of section 2).

Our next step is to turn on the background B field in order to study the tilted bion from the noncommutative D1-brane point of view. The only difference from the previous case is that we turn on the component $B_{12} = \text{const.}$ of the background B field (remember that the D1-brane worldvolume spans the (X^0, X^9) plane and that only the $i = 1, 2, 3$ components of the matrix transverse displacement field are allowed to be nonzero). It is still the case that the action is a functional only of the fields $\partial\Phi^i$ and $\epsilon_{ijk}[\Phi^j, \Phi^k]$ and the same reasoning as before leads us to treat these quantities as commuting objects inside the STr instruction. In this way, the action (1.1) can be reduced to something much more explicit. To get the most transparent results, it helps to define rescaled fields

$$\varphi_1 = \sqrt{(1 + \lambda^2 B^2)}\Phi_1\,, \qquad \varphi_2 = \sqrt{(1 + \lambda^2 B^2)}\Phi_2\,, \qquad \varphi_3 = \Phi_3\,, \tag{3.16}$$

and to redefine the commutator W^i as

$$W_i \equiv \frac{1}{2} i\epsilon_{ijk}\left[\varphi^j, \varphi^k\right] - \delta_i^3 B\,. \tag{3.17}$$

After some rather tedious algebra (made much easier by MAPLE) to evaluate the determinants in the definition of the action, we get a result for the energy functional that is almost identical to (3.3):

$$\begin{aligned} U_{B\neq 0} &= \frac{1}{\sqrt{1+\lambda^2 B^2}} \int d\sigma \, \mathrm{STr} \sqrt{1+\lambda^2(\partial\varphi^i)^2+\lambda^2(W_i)^2+\lambda^4(\partial\varphi^i W_i)^2} \\ &= \frac{1}{\sqrt{1+\lambda^2 B^2}} \int d\sigma \, \mathrm{STr} \sqrt{(1\pm\lambda^2\partial\varphi^i W_i)^2+\lambda^2(\partial\varphi^i \mp W_i)^2}\,. \end{aligned} \tag{3.18}$$

The action is still 'linearized' by taking $\partial\varphi^i = \pm W_i$, which means that the BPS condition in the presence of a background B field is

$$\partial\varphi_i = \pm i\left(\frac{1}{2}\epsilon_{ijk}\left[\varphi^j,\varphi^k\right] + \delta_i^3 iB\right). \tag{3.19}$$

This is precisely the generalization of the Nahm equation that has been derived in the context of studies of magnetic monopoles in noncommutative field theory [12]. It is a plausible candidate for the exact BPS condition for the nonabelian D1-brane system and we will show that it gives a detailed account of the physics of the tilted bion. The fact that the generalized Nahm equation is implied by the Myers action is further evidence for the essential correctness of the latter.

In order to solve the modified equations of motion, we have to slightly modify the ansatz (3.6), expressing the fields φ in terms of generators of an N-dimensional representation of SU(2) and a scalar function $\hat{R}(\sigma)$:

$$\varphi^i = \hat{R}(\sigma)\alpha^i - \delta_i^3 \frac{B}{2\hat{R}(\sigma)}\,. \tag{3.20}$$

When this modified ansatz is substituted into the BPS equation (3.19), we obtain the same equation for $\hat{R}$ as before, namely $\hat{R}' = \mp 2\hat{R}^2$: solution (3.8) still holds. If we collect the generators into a modified triplet $\mathbf{X} \equiv (\alpha^1, \alpha^2, \alpha^3 + \frac{B}{2\hat{R}^2})$ and use (3.20), the action (3.18) can be expressed as an effective action for $\hat{R}(\sigma)$:

$$U_{B\neq 0}[\hat{R}(\sigma)] = \frac{1}{\sqrt{1+\lambda^2 B^2}} \int d\sigma \, \mathrm{STr} \sqrt{(1+\lambda^2(\hat{R}')^2\mathbf{X}^2)(1+4\lambda^2(\hat{R})^4\mathbf{X}^2)}\,. \tag{3.21}$$

This looks the same as (3.9) but is not quite because $\mathbf{X}$ now depends on $\hat{R}$. Nevertheless, the same radial function (3.8) continues to be a solution, further testing the compatibility of the action with the BPS condition. Notice that this does not conclusively prove that the ansatz (3.20) with (3.8) is a solution to the full equations of motion implied by (3.18).

It is easily seen that this solution corresponds to the tilted bion solution discussed in the previous section. Equation (3.16) matches the ratios of axes given in (2.7). The over-all size of the spheroid agrees with (2.7) by an argument identical to that given for $B = 0$. The shift of its center is given by

$$\Delta^i(\sigma) = \frac{1}{N} tr(\lambda\Phi^i) = \Delta\delta_3^i \tag{3.22}$$

JHEP05(2002)038

(by virtue of the fact that $tr(\alpha^i) = 0$), where

$$\Delta = -\frac{\lambda B}{2\hat{R}} = \mp\lambda B\sigma = \begin{cases} \tan(\alpha)|\sigma| & \text{for cases } \mathbf{A} \text{ and } \mathbf{D}. \\ -\tan(\alpha)|\sigma| & \text{for cases } \mathbf{B} \text{ and } \mathbf{C}. \end{cases} \tag{3.23}$$

Thus, in cases **A** and **D**, the bion tilts in agreement with section 2. In the other two cases, it tilts in the opposite direction. The interpretation is that D1-branes coming 'towards' the D3-brane correspond to a positively charged magnetic monopole, while D1-branes coming 'away from' the D3-brane correspond to a negatively charged one. Similarly, $\overline{\text{D1}}$-strings coming 'away from' the D3-brane correspond to a positively charged magnetic monopole, while D1-branes coming 'towards' the D3-brane correspond to a negatively charged one. The geometry of the tilted bion inferred from the nonabelian dynamics of D1-branes perfectly matches the results of the abelian D3-brane calculation summarized in the previous section.

4. Flat D$(p+r)$-brane from Dp-branes

It is known that within Yang-Mills theory, lower dimensional branes can expand to form higher dimensional noncommutative branes (see, for example, [7] and references therein). In this section, we show how this construction can be extended to the nonlinear case of the nonabelian BI action. The point of this exercise (which looks like a detour from the line of argument of the rest of the paper) is to infer a specific recipe for evaluating the worldvolume gauge fields in the noncommutative description of a D-brane. In the next section we will apply the spirit of this recipe to the curved branes which are our primary interest.

We take the spacetime metric to be the flat Minkowski metric ($g_{\mu\nu} = \eta_{\mu\nu}$), the dilaton to be constant and the worldvolume gauge field to be zero. We take the background two-form field B to have nonzero (constant) components only in directions transverse to the brane, $i, j, k = p+1, \ldots, 9$. The world-volume of the branes is parametrized by $X^a = \sigma^a, a = 0, \ldots, p$ (i.e. we are using the static gauge) and the transverse fluctuations are $X^i = \lambda\Phi^i$, where Φ^i are $N \times N$ matrices in the adjoint of the gauge group.

We will look for solutions where the transverse scalars Φ are not functions of the brane coordinates σ^a. In this case, the action (1.1) for the nonabelian dynamics of N Dp-branes reduces to

$$S = -gT_p \int d^p x^a \, \mathrm{STr}\left(\sqrt{\det(Q^i_j)}\right) . \tag{4.1}$$

where the explicit form of Q is displayed in (1.2). It is easy to to show that matrices Φ^i, satisfying

$$[\Phi^i, \Phi^j] = \frac{i}{\lambda^2}\theta^{ij} I_{N\times N} , \tag{4.2}$$

where θ^{ij} is an arbitrary $(9-p) \times (9-p)$ antisymmetric matrix of c-numbers, solve the equations of motion. The question that this equation cannot be solved for finite N will be addressed shortly.

Substituting this solution into the general definition of Q, (1.2), gives

$$Q^i_j = \delta^i_j - \lambda^{-1}\theta^{il}(g + \lambda B)_{lj}\,. \tag{4.3}$$

θ can have any even rank r up to $9-p$. With no loss of generality, we can block diagonalize θ so that $\theta^{\mu\nu} \neq 0$ for $\mu, \nu = p+1, \ldots, p+r$. The remaining directions will be denoted by $m, n = p+r+1, \ldots, 9$ and of course $\theta^{in} = 0$ for i, n in their appropriate ranges. From now on, $\theta = \theta^{\mu\nu}$ will denote an invertible, $r \times r$ matrix with inverse $\theta_{\mu\nu}$. Also, let us restrict our attention to background two-form fields B only in directions $p+1, \ldots, p+r$, since the other components can simply be gauged away.

The above can be summarized by saying that solution (4.2) divides the Φ^i's into $r/2$ pairs satisfying canonical commutation relations and $9-p-r$ other commuting coordinates (which we can drop from further consideration). The slight hitch is that canonical commutation relations can only be realized on infinite-dimensional function spaces, and not on finite-dimensional matrices. The solution only makes sense if we take $N \to \infty$ and reinterpret all matrix operations (multiplication, trace, etc.) in the action as the corresponding Hilbert space operations. Fortunately, the technology for doing this sort of thing has been worked out in the study of noncommutative field theory over the past couple of years. Indeed, the physics of small fluctuations about (4.2) is best described by a noncommutative field theory on the r-dimensional base space spanned by the noncommuting Φ^i. It is in this sense that we will interpret the existence of r noncommuting transverse displacement fields on the Dp-brane as creating an effective D$(p+r)$-brane. We will first show that the energetics of (4.2) are indistinguishable from that of an abelian D$(p+r)$-brane with a certain worldvolume gauge field (which is the quantity that is the focus of our interest).

To establish the desired result, we make use of an equivalence established in noncommutative field theory between actions built on ordinary integrals of functions of ordinary coordinates, but with a noncommutative definition of multiplication of functions (the $*$-product or Moyal product) and actions where functions become operators on a harmonic oscillator Hilbert space and the action is computed as the trace of an operator on that Hilbert space (integral over space becomes Hilbert space trace) [13, 14]. In our interpretation of (4.2), the STr operation in the action (4.1) is to be thought of as a trace over operators on a Hilbert space. As just indicated, the trace can be recast as an integral over the associated noncommuting coordinates, but we need the exact relative normalization. The identification we need has been worked out in the noncommutative field theory literature [7]:

$$d^r x^\mu \longleftrightarrow (2\pi)^{\frac{r}{2}} \mathrm{Pf}(\theta)\, \mathrm{STr}\,, \tag{4.4}$$

where Pf denotes the Pfaffian: $\mathrm{Pf}(\theta)^2 = \det(\theta)$. Putting the various pieces of the puzzle together, we can express the action for our solution in the form of an equivalent integral of an energy density over a D(r+p)-brane worldsheet:

$$\begin{aligned} S(\theta) &= -gT_p \int d^p x^a d^r x^\mu \frac{1}{\sqrt{\det(2\pi\theta)}} \left(\sqrt{\det(\delta^i_j - \lambda^{-1}\theta^{il}(g+\lambda B)_{lj})}\right) \\ &= -gT_p (2\pi\lambda)^{-\frac{r}{2}} \int d^p x^a d^r x^\mu \left(\sqrt{\det(g + \lambda B - \lambda\theta^{-1})}\right). \end{aligned} \tag{4.5}$$

Since $T_{p+r} = T_p(2\pi\lambda)^{-r/2}$, the object we have constructed has the right energy density to be a D$(p+r)$-brane with a world-volume F-flux equal to $-\theta^{-1}$. Noncommutative coordinates for a lower-dimensional brane have, in a simple context, been converted to a higher dimensional brane carrying a worldvolume gauge field. For future reference, the interesting thing is the way the gauge field arises via the commutator of the lower-dimensional matrix coordinates (4.2).

To give this idea a more demanding test, we will now check whether we can reproduce the correct Chern-Simon couplings. We will need the nonabelian Chern-Simons action proposed by Myers in [1]

$$S_{CS} = \mu_p \int \mathrm{STr}\left(P\left[e^{i\lambda i_\Phi i_\Phi}(\Sigma C^{(n)} e^{\lambda B})\right] e^{\lambda F}\right) \tag{4.6}$$

(we refer the reader to Myers' paper for the definition of the symbol $\lambda i_\Phi i_\Phi$ and other notation). For concreteness, specialize to p=1 (nonabelian D-strings extending in the 01-directions), with θ and B nonzero only in the 23-directions. Examine the coupling to $C^{(2)}$, specifically, to the C_{01} component. Expand the action (4.6) and pick off the coupling of interest to obtain

$$S = \mu_1 \int dx^0 dx^1 \, \mathrm{STr}\, C_{01}\left((1 + (i\lambda i_\Phi i_\Phi)\lambda B\right). \tag{4.7}$$

Using the solution (4.2) for Φ, we have $(i\lambda i_\Phi i_\Phi)\lambda B = i\lambda^2[\Phi^1, \Phi^2]B_{12} = -\theta B$. Passing from STr to $\int$ according to (4.4) we obtain an expression for this interaction in terms of an integral over the D$(p+r)$-brane worldvolume:

$$S = \mu_3 \int dx^0 dx^1 dx^2 dx^3 C_{01}\left(B - \theta^{-1}\right). \tag{4.8}$$

This is precisely the right coupling for a D3-brane with a world-volume field $F_{23} = -\theta^{-1}$. The world-volume field F suggested here corresponds precisely to the field which one would expect to get from the D-strings dissolved in a D3-brane with density θ^{-1} per area normal to the D-strings. Solution (4.2) corresponds to exactly this density of D-strings.

The lesson we learn from this computation is that when Dp-branes expand to form a D$(p+r)$-brane, the world volume gauge field F on the D$(p+r)$-brane can be computed from the inverse of the density of Dp-branes, which in turn can be obtained from the commutators of the transverse coordinates. Notice that we compute here F and not $F+B$. F lives in the worldvolume of the resuling D$(p+r)$-brane and is the result of a dissolved density of Dp-branes, while B lives in the bulk and is part of the input to the action of the Dp-branes. The gauge symmetry which exchanges F and B on the D$(p+r)$-brane would have to be realised in the Dp-brane scenario as a symmetry involving both B and the transverse coordinates Φ^is. It is not clear what this symmetry is.

5. The worldvolume gauge field on the dual bion

In this section, we will take the prescription given in section 4 for identifying the world-volume gauge field implicit in a set of noncommuting coordinates and adapt it to the bion problem under discussion.

We begin with the simple example of N D1-branes (N large) with $B = 0$ (the setup of section 3). Choose the solution (3.6) based on the $N \times N$ representation of SU(2) and further specialize to the representation where α^3 is diagonal: $\alpha^3 = \mathrm{diag}(N-1, N-3, N-5, \ldots, -N+3, -N+1)$. The sphere described by equation (3.6) at a fixed σ goes through a point $(X^1, X^2, X^3) = (0, 0, R = \pi\alpha' N/|\sigma|)$. A small patch of the sphere near this point is described by the corner $k \times k$ blocks of the full SU(2) representation matrixes α^i, where $k \ll N$. Explicitly, replace α^3 with $\mathrm{diag}(N-1, N-3, \ldots, N-2k+1)$, which for $k \ll N$ is approximately just $N\, I_{k\times k}$. The small patch of the sphere is now described by the same commutator as the noncommutative plane (section 4),

$$\left[\Phi^1, \Phi^2\right] = i(2\hat{R})\Phi^3 \to \pm i \frac{N}{2\sigma^2}\,. \tag{5.1}$$

The '+'-sign corresponds to cases **B** and **C**, and the '−'-sign corresponds to cases **A** and **D** in section 3. Following section 4, we now write

$$\theta^{12} = -i\lambda^2 \left[\Phi^1, \Phi^2\right] = \frac{\lambda^2 N}{2\sigma^2}\,, \tag{5.2}$$

so that the identification $F = -\theta^{-1}$ gives

$$F_{12} = \pm\frac{2\sigma^2}{\lambda^2 N} = \pm\frac{N}{2R^2}\,. \tag{5.3}$$

Referring back to (2.9) in section 2, we see that this is indeed the correct value of the worldvolume gauge field. Again, in cases **A** and **D**, we obtain the '−'-sign while in cases **B** and **C**, we obtain the opposite sign and monopole charge. This is all in agreement with expectations from (2.9). The essence of this computation is that the commutator $[\Phi^i, \Phi^j]$ defines a two-form field in the worldvolume of the D3-brane, whose inverse is the worldvolume gauge field F.

Computing the worldvolume gauge field in case of $B \neq 0$ is very similar, except the geometry is more complicated. To avoid any formulas with multiple $\pm$ signs, we will specialize to case **A** above, choosing $\sigma < 0$ and $\hat{R} = (2\sigma)^{-1}$ (the other three cases are similar). We want to evaluate the gauge field on the tilted D3-brane implied by the nonabelian solution (3.20) and compare it to the result of a direct calculation given in (2.9). To do this, it is best to convert (3.20) to the coordinates given in (2.4). Defining rotated variables

$$\Psi^1 = \Phi^1\,, \quad \Psi^2 = \Phi^2\,, \quad \Psi^3 = \cos(\alpha)\Phi^3 - \sin(\alpha)\sigma/\lambda\,, \quad \Psi^4 = \sin(\alpha)\Phi^3 + \cos(\alpha)\sigma/\lambda \tag{5.4}$$

and then inserting (3.20) gives

$$\begin{aligned}
\Psi^1 &= \cos(\alpha)\frac{1}{2\sigma}\,\alpha^1\,,\\
\Psi^2 &= \cos(\alpha)\frac{1}{2\sigma}\,\alpha^2\,,\\
\Psi^3 &= \cos(\alpha)\frac{1}{2\sigma}\,\alpha^3\,,\\
\Psi^4 &= \sin(\alpha)\frac{1}{2\sigma}\,\alpha^3 + \frac{1}{\cos(\alpha)}\,\frac{\sigma}{\lambda}\,.
\end{aligned} \tag{5.5}$$

To check that this makes sense, take N large and pass to the classical limit, by setting $\alpha^i \to N n^i$ where $(n^1)^2 + (n^2)^2 + (n^3)^2 = 1$. The Ψ's become classical coordinates

$$
\begin{aligned}
\lambda\Psi^1 &\to Y^1 = \cos(\alpha)\frac{N\pi\alpha'}{\sigma}\, n^1 \,, \\
\lambda\Psi^2 &\to Y^2 = \cos(\alpha)\frac{N\pi\alpha'}{\sigma}\, n^2 \,, \\
\lambda\Psi^3 &\to Y^3 = \cos(\alpha)\frac{N\pi\alpha'}{\sigma}\, n^3 \,, \\
\lambda\Psi^4 &\to Y^4 = \sin(\alpha)\frac{N\pi\alpha'}{\sigma}\, n^3 + \frac{1}{\cos(\alpha)}\,\sigma \\
& = \tan(\alpha) Y_3 + \frac{N\pi\alpha'}{\sqrt{Y_1^2 + Y_2^2 + Y_3^3}}\,,
\end{aligned}
\tag{5.6}
$$

in perfect correspondence with equation (2.5).

In section 4 we showed that the worldvolume gauge field is computed from the commutators of the transverse scalars. Using (5.5) to compute the commutators and comparing with (4.2), we obtain the following noncommutativity tensor Θ:

$$
\begin{aligned}
-i\lambda^2[\Psi^i, \Psi^j] &= 2\epsilon^{ijk}\frac{\lambda^2\cos(\alpha)}{2\sigma}\Psi^k \to \frac{\lambda\cos(\alpha)}{\sigma}\epsilon^{ijk} Y^k \equiv \Theta^{ij}\,, \\
-i\lambda^2[\Psi^1, \Psi^4] &= -2\frac{\lambda^2\sin(\alpha)}{2\sigma}\Psi^2 \to -\frac{\lambda\sin(\alpha)}{\sigma} Y^2 \equiv \Theta^{14}\,, \\
-i\lambda^2[\Psi^2, \Psi^4] &= 2\frac{\lambda^2\sin(\alpha)}{2\sigma}\Psi^1 \to \frac{\lambda\sin(\alpha)}{\sigma} Y^1 \equiv \Theta^{24}\,, \\
-i\lambda^2[\Psi^3, \Psi^4] &= 0 \to 0 \equiv \Theta^{34}\,,
\end{aligned}
\tag{5.7}
$$

where $i, j, k = 1, \ldots, 3$. Finally, we need to pull the two-tensor Θ back to the worldvolume of the D3-brane, to the worldvolume coordinates of (2.5). With a little algebra, it can be checked that $\Theta^{\mu\nu}$ $(\mu, \nu = 1, \ldots, 4)$ satisfies

$$
\Theta^{\mu\nu} = \frac{\partial Y^\mu}{\partial y^i}\frac{\partial Y^\nu}{\partial y^j}\theta^{ij}\,,
\tag{5.8}
$$

where

$$
\theta^{ij} = \frac{\lambda\cos(\alpha)}{\sigma}\,\epsilon^{ijk} y^k = -\frac{2\sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}}{N}\,\epsilon^{ijk} y^k
\tag{5.9}
$$

(the minus sign is a consequence of our having chosen case **A**, $\sigma < 0$). According to section 4, the worldvolume gauge field is the negative inverse of the noncommutativity tensor θ^{ij}. However, the tensor of (5.9) is not invertible: it acts in three dimensions and has one zero eigenvalue. Following section 4, the inversion of θ^{ij} is to be carried out on the subspace orthogonal to the subspace of zero eigenvalues. With this understanding, we obtain the following result for the gauge field on the D3-brane

$$
F_{ij} = (-\theta^{-1})_{ij} = -\frac{N}{2\left[(y^1)^2 + (y^2)^2 + (y^3)^2\right]^{3/2}}\,\epsilon^{ijk} y^k\,,
\tag{5.10}
$$

in perfect agreement with the abelian D3-brane result, equation (2.9). This is what we wanted to show. Notice that we compute here F and not $F + B$. F is the result of a dissolved density of D1-branes in the D3-brane, while B lives in the bulk and is part of the input to the action of the D1-branes.

6. Conclusion

We have shown that the noncommutative bion solution from [4] can be generalized to include a nonzero NS-NS two-form field B. The geometry extracted from our generalized solution agrees with the 'dual' picture provided by the abelian theory on a D3-brane in the presence of a nonzero B. Even better, we have been able to argue that the nonabelian calculation makes a prediction for the worldvolume gauge field on the D3-brane and we find that this agrees with the abelian caluculation as well. Although limited to BPS configurations, we regard these considerations as a significant further test of the validity of the nonabelian Born-Infeld action proposed by Myers in [1].

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